3 a) $A > 0$ means that $x^T A x > 0$ for all nonzero vectors $x \in \mathbb{R}^n$. Let $k \in \{1, 2, \ldots, n\}$.
Also let $y \in \mathbb{R}^k$ be an arbitrary nonzero vector.

$$0 < \begin{pmatrix} y \\ 0 \end{pmatrix}^T A \begin{pmatrix} y \\ 0 \end{pmatrix} = y^T A_k y.$$

So $A_k > 0$.

During the lectures, we have seen that $A_k > 0$ if and only if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of A_k are positive.
Thus, $\det(A_k) = \lambda_1 \cdot \lambda_2 \cdots \lambda_k > 0$.

b) $A_i > 0$ implies that all eigenvalues of A_i are positive.
Since the eigenvalues of

$$P := \begin{bmatrix} A_i & 0 \\ 0 & a_i - b_i^T A_i^{-1} b_i \end{bmatrix}$$

are the union of the eigenvalues of A_i and $a_i - b_i^T A^{-1} b_i > 0$, the eigenvalues of P are positive. Thus, P is positive definite.

Now, note that

$$M = \begin{bmatrix} I & A_i^{-1} b_i \\ 0 & 1 \end{bmatrix}$$

is nonsingular (it has determinant 1).

Therefore, if $x \in \mathbb{R}^n$ is nonzero, also $Mx \neq 0$.

In addition,

$$M^T = \begin{pmatrix} I & 0 \\ b_i^T A_i^{-1} & 1 \end{pmatrix},$$ since A_i is symmetric.

Hence, for any nonzero $x \in \mathbb{R}^n$, $x^T A_{i+1} x =$
$x^T M^T P \underbrace{Mx}_{\neq 0} > 0$, by positive definiteness of P.

As such, $A_{i+1} > 0$.

c) Assume that $\det(A_k) > 0$ for $k = 1, 2, \ldots, n$.
We will prove that $A_k > 0$ for $k = 1, 2, \ldots, n$, thus $A = A_n > 0$.

① $A_1 = \det(A_1) > 0$ ✓

② Assume that $A_i > 0$ for some $i \in \{1, 2, \ldots, n-1\}$.
To prove: $A_{i+1} > 0$.

$$A_{i+1} = M^T \begin{bmatrix} A_i & 0 \\ 0 & a_i - b_i^T A_i^{-1} b_i \end{bmatrix} M.$$

$$\det(A_{i+1}) = \underbrace{\det(M^T)}_{=1} \cdot \det(P) \cdot \underbrace{\det(M)}_{=1}$$

while $\det(P) = \det(A_i) \cdot \det(a_i - b_i^T A_i^{-1} b_i)$

so $\underbrace{\det(A_{i+1})}_{>0} = \underbrace{\det(A_i)}_{>0} \cdot \det(a_i - b_i^T A_i^{-1} b_i)$.

$\Rightarrow \det(a_i - b_i^T A_i^{-1} b_i) = a_i - b_i^T A_i^{-1} b_i > 0$
$\Rightarrow A_{i+1} > 0$ by (b).

Therefore, by induction, $A_k > 0$ for $k = 1, 2, \ldots, n$.
Hence, $A = A_n > 0$.

d) $1 > 0$, $\det \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = 2 > 0$

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 1 > 0.$$

So A is positive definite by (c).

4 a)

$$A = \begin{bmatrix} a & -b \\ b & a \\ c & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} a^2+b^2+c^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$$

so eigenvalues of A^TA are $a^2+b^2+c^2 > a^2+b^2$.

$\Rightarrow$ $\sigma_1 = \sqrt{a^2+b^2+c^2}$ and $\sigma_2 = \sqrt{a^2+b^2}$.

note that $\sigma_1 > 0$ and $\sigma_2 > 0$, since $a, b, c \neq 0$.

A corresponding orthogonal matrix V is:

$$V = \begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ v_1 \quad v_2 \end{array}.$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$$

Extend u_1, u_2 to an orthonormal basis of $\mathbb{R}^3$:

For example, choose $u_3 = \dfrac{1}{\sqrt{a^2+b^2+\frac{1}{c^2}(a^2+b^2)^2}} \begin{pmatrix} a \\ b \\ \frac{1}{c}(-a^2-b^2) \end{pmatrix}$

Define $U = \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2+c^2}} & \frac{-b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2+\frac{1}{c^2}(a^2+b^2)^2}} \\ \frac{b}{\sqrt{a^2+b^2+c^2}} & \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2+\frac{1}{c^2}(a^2+b^2)^2}} \\ \frac{c}{\sqrt{a^2+b^2+c^2}} & 0 & \frac{-a^2-b^2}{c\sqrt{a^2+b^2+\frac{1}{c^2}(a^2+b^2)^2}} \end{pmatrix}$

By construction, U is an orthogonal matrix.

Define $\Sigma = \begin{pmatrix} \sqrt{a^2+b^2+c^2} & 0 \\ 0 & \sqrt{a^2+b^2} \\ 0 & 0 \end{pmatrix}$

Then $A = U\Sigma V^T$.

b) A best rank-1 approximation is

$$X = U \begin{pmatrix} \sqrt{a^2+b^2+c^2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} V^T$$

$$= \begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \end{pmatrix}$$

The distance of A to the set of 3×2 matrices of rank ≤ 1 is:

$$d(A, M_1) = \sigma_2 = \sqrt{a^2+b^2}$$